

# Nonrelativistic Quantum Hamiltonian for Lorentz Violation

V. Alan Kostelecký and Charles D. Lane

*Physics Department, Indiana University, Bloomington, IN 47405, U.S.A.*  
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A method is presented for deriving the nonrelativistic quantum hamiltonian of a free massive fermion from the relativistic lagrangian of the Lorentz-violating standard-model extension. It permits the extraction of terms at arbitrary order in a Foldy-Wouthuysen expansion in inverse powers of the mass. The quantum particle hamiltonian is obtained and its nonrelativistic limit is given explicitly to third order.

## I. INTRODUCTION

Establishing the physical relevance of a lagrangian in relativistic quantum field theory often requires a determination of its nonrelativistic content. The Foldy-Wouthuysen (FW) transformation [1] provides a systematic approach to understanding the low-energy effects of certain theories. Given the relativistic quantum hamiltonian for a theory of massive four-component fermions, the nonrelativistic quantum hamiltonian for the corresponding two-component particle can be derived in an expansion in inverse powers of the fermion mass.

In this work, we use generalized FW methods to investigate the quantum particle hamiltonian that describes the physics of a free massive two-component fermion emerging from the relativistic lagrangian of the Lorentz-violating standard-model extension [2]. This standard-model extension is based on the idea of spontaneous Lorentz breaking in an underlying theory [3] and has been used for various investigations placing constraints on possible violations of Lorentz symmetry [2,4–15], several of which depend crucially on the nonrelativistic physics of free massive fermions. In these investigations, specific terms in the nonrelativistic hamiltonian have been derived as needed, but a full treatment has been lacking. Here, we provide a systematic approach that permits extraction of the relevant terms in the nonrelativistic hamiltonian at arbitrary order in the FW approximation. We obtain the quantum particle hamiltonian and provide explicitly the form of the nonrelativistic hamiltonian to third order. Our results are directly relevant to recent analyses of muon and clock-comparison experiments [14,15] and are expected to have substantial impact on further studies of the physical implications of the standard-model extension.

The general form of the relativistic lagrangian for a free spin- $\frac{1}{2}$  Dirac fermion  $\psi$  of mass  $m$  in the standard-model extension is [2]

$$\mathcal{L} = \frac{1}{2}i\bar{\psi}(\gamma_\nu + c_{\mu\nu}\gamma^\mu + d_{\mu\nu}\gamma_5\gamma^\mu + e_\nu + if_\nu\gamma_5 + \frac{1}{2}g_{\lambda\mu\nu}\sigma^{\lambda\mu})\partial^\nu\psi - \bar{\psi}(m + a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu + \frac{1}{2}H_{\mu\nu}\sigma^{\mu\nu})\psi \quad (1)$$

This is a generalization of the usual relativistic lagrangian for a free massive Dirac fermion. The Dirac matrices  $\{1, \gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}\}$  have conventional properties, and the Minkowski metric  $\eta_{\mu\nu}$  has signature  $-2$ . The parameters  $a_\mu, b_\mu, c_{\mu\nu}, d_{\mu\nu}, e_\mu, f_\mu, g_{\lambda\mu\nu}$ , and  $H_{\mu\nu}$  control the extent of Lorentz violation in the theory. In a given observer inertial frame, they can be regarded as fixed real Lorentz vectors or tensors. Note that  $H_{\mu\nu}$  can be taken as antisymmetric,  $c_{\mu\nu}$  and  $d_{\mu\nu}$  as traceless, and  $g_{\lambda\mu\nu}$  as antisymmetric in the first two indices. Since Lorentz symmetry is known to be valid to high precision, any nonzero parameters in nature would need to be minuscule. We therefore restrict our attention in this work to terms linear in these parameters.

In the next section, the relativistic particle-antiparticle hamiltonian  $H$  corresponding to the lagrangian (1) is obtained. Some basic information about our procedure for extracting its FW form is discussed in section III, together with our definition of the relevant FW sequence. Features of this sequence are derived in section IV, and the quantum particle hamiltonian and its nonrelativistic limit to third order are explicitly presented in section V.

## II. RELATIVISTIC QUANTUM HAMILTONIAN

The first step in deriving low-energy effects of the Lorentz-violating terms is to obtain the relativistic hamiltonian  $H$  associated with the lagrangian (1). However, methods for direct construction of  $H$  are inadequate because Eq. (1) contains couplings involving time derivatives. For example, applying the Euler-Lagrange equations to  $\mathcal{L}$  and solving for  $H$  from the equation  $i\partial_0\psi = H\psi$  results in a non-hermitian hamiltonian and a corresponding nonunitary time evolution.

One method of bypassing this technical difficulty is to perform a field redefinition  $\psi = A\chi$  in the lagrangian [5], with  $A$  chosen such that the dependence of the lagrangian on  $\partial_0\chi$  is just that of the usual Dirac lagrangian. Then, the wave function associated with  $\chi$  evolves conventionally in time. The field redefinition leaves unchanged the physics, while it causes the time-derivative couplings to be replaced by extra terms in the lagrangian.

To implement this idea, we write the lagrangian (1) in the forms

$$\begin{aligned}\mathcal{L} &\equiv \frac{1}{2}i\bar{\psi}\Gamma_\nu \overset{\leftrightarrow}{\partial}^\nu \psi - \bar{\psi}M\psi \\ &= \frac{1}{2}i\bar{\chi}\gamma_0 \overset{\leftrightarrow}{\partial}^0 \chi + \frac{1}{2}i\bar{\chi}(\bar{A}\Gamma_j A) \overset{\leftrightarrow}{\partial}^j \chi - \bar{\chi}(\bar{A}MA)\chi \quad ,\end{aligned}\quad (2)$$

where  $\Gamma_\nu$  and  $M$  are defined according to the correspondence with Eq. (1), and  $\bar{\psi} = \bar{\chi}\bar{A}$  with  $\bar{A} := \gamma^0 A^\dagger \gamma^0$ . In the second expression the Lorentz indices are separated into timelike and spacelike cartesian components,  $\mu \equiv 0$  and  $j = 1, 2, 3$ , with summation on repeated indices understood.

The choice

$$A = 1 - \frac{1}{2}\gamma^0(\Gamma_0 - \gamma_0) \quad , \quad \bar{A} = 1 - \frac{1}{2}(\Gamma_0 - \gamma_0)\gamma^0 \quad (3)$$

implements the equality (2) to linear order in the parameters for Lorentz violation. Derivation of the relativistic hamiltonian  $H$  can then proceed through the Euler-Lagrange equations, which take the form of a modified Dirac equation:

$$(i\bar{A}\Gamma_\mu A\partial^\mu - \bar{A}MA)\chi = 0 \quad . \quad (4)$$

We find

$$H = -\gamma^0\bar{A}\Gamma_j A p^j + \gamma^0\bar{A}MA \quad , \quad (5)$$

where the three-momentum of the particle is denoted  $p_j$ , and  $H$  obeys the equation  $i\partial_0\chi = H\chi$ .

Explicitly, the relativistic hamiltonian can be written

$$H = m(\gamma^0 + \mathcal{P}_0 + \mathcal{O}_0 + \mathcal{E}_0) \quad , \quad (6)$$

where

$$\begin{aligned}m\mathcal{P}_0 &:= -p_j\gamma^0\gamma^j \quad , \\ m\mathcal{O}_0 &:= [-b_0 + (d_{0j} + d_{j0})p^j]\gamma_5 + [a_j - (c_{jk} - c_{00}\eta_{jk})p^k]\gamma^0\gamma^j + if_j p^j \gamma_5 \gamma^0 + i[H_{0j} + (g_{j0k} + g_{jk0})p^k]\gamma^j \quad , \\ m\mathcal{E}_0 &:= [a_0 - (c_{0j} + c_{j0})p^j - me_0] + [-b_j + (d_{jk} - d_{00}\eta_{jk})p^k - \frac{1}{2}m\varepsilon^{klm}\eta_{jm}g_{kl0}]\gamma_5\gamma^0\gamma^j \\ &\quad - [mc_{00} + e_j p^j]\gamma^0 - [\frac{1}{2}\varepsilon^{klm}\eta_{jm}H_{kl} + md_{j0} - \varepsilon^{lmn}\eta_{jn}(\frac{1}{2}g_{lmk} - \eta_{km}g_{l00})p^k]\gamma_5\gamma^j \quad .\end{aligned}\quad (7)$$

In these expressions, the totally antisymmetric rotation tensor  $\varepsilon^{jkl}$  satisfies  $\varepsilon_{123} = +1$  and  $\varepsilon^{jkl} = -\varepsilon_{jkl}$ . The particular decomposition of  $H$  into the four terms in (6) is chosen for later convenience.

As an aside, we remark that the relativistic hamiltonian is also readily found if the theory (1) is extended to include a minimal coupling to a U(1) gauge field  $A_\mu$ . It suffices to replace the partial derivative  $i\partial_\mu$  in Eq. (1) with the covariant derivative  $iD_\mu := i\partial_\mu - qA_\mu$ , where  $q$  is the particle charge. The relativistic hamiltonian then has the same form as in Eqs. (6) and (7) above, except that all occurrences of  $p_j$  must be replaced with  $\pi_j := p_j - qA_j$  and the term  $qA_0$  must be added to Eq. (6). The resulting hamiltonian is relevant, for example, for studies of Lorentz-violating effects in quantum electrodynamics.

### III. DEFINITION OF THE FW SEQUENCE

In the strict nonrelativistic limit, the lower two components of the relativistic wave function  $\chi$  are negligible, so the upper two components of  $\chi$  suffice to determine the nonrelativistic particle behavior. However, more generally the Dirac equation couples the upper and lower components of  $\chi$ . The object of the Foldy-Wouthuysen procedure is to find a (momentum-dependent) unitary transformation

$$H \mapsto \tilde{H} := e^{iS} H e^{-iS} = \exp[\text{ad}(iS)]H \quad , \quad (8)$$

where  $\text{ad}(X)Y := [X, Y]$ , such that  $\tilde{H}$  is  $2 \times 2$  block diagonal. This therefore decouples the upper and lower components of the FW-transformed wave function  $\phi := e^{iS}\chi$ . Requiring hermiticity of  $S$  ensures that  $e^{iS}$  is unitary. It follows that  $\tilde{H}$  is hermitian and that both hamiltonians  $H$  and  $\tilde{H}$  describe the same physics. The FW transformation amounts to a unitary rotation in the Hilbert space of the free particle states that preserves the dominance of the upper two components of the wave function. The quantum particle hamiltonian  $h_{\text{rel}}$  and the nonrelativistic limit  $h$  we seek are given by the leading  $2 \times 2$  block of  $\tilde{H}$ .

Solving directly for  $\tilde{H}$  would be of interest but is challenging in the general case. Instead, we present a method that allows approximation of  $\tilde{H}$  to arbitrary accuracy in an expansion in powers of  $|\vec{p}|/m$ . The basic idea is to apply a succession of transformations of the type (8), chosen so that each iteration of the transformed hamiltonian has a smaller block off-diagonal part than the previous one. The exact FW transformation is the limit of this sequence. Although more direct approaches can yield a low-order approximation to  $h$  without the use of our method, the results derived here permit straightforward calculation of  $h_{\text{rel}}$  and of  $h$  to any desired order.

For definiteness in what follows, we work within the Dirac-Pauli representation of the Dirac matrices, for which  $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$ , where  $\sigma^j$  are the usual Pauli matrices. We define a matrix to be *even* if it is block diagonal and *odd* if it is block off-diagonal. Any  $4 \times 4$  matrix  $X$  can be uniquely written as the sum of an even part and an odd part,  $X = \text{Even}(X) + \text{Odd}(X)$ , where  $\text{Odd}(X) = \frac{1}{2}\gamma^0[\gamma^0, X]$  and  $\text{Even}(X) = \frac{1}{2}\gamma^0\{\gamma^0, X\}$ .

We seek a sequence of FW transformations such that the odd part of the hamiltonian progressively decreases in some suitable matrix norm, such as  $\|A\| := \max_{a,b}\{|A_{ab}|\}$  for  $a, b = 1, 2, 3, 4$ . In the remainder of this section, an appropriate sequence  $\{H_n\}$  of hamiltonians is introduced. For each  $n$ , we also introduce a parameter  $t_n$  that turns out to provide a measure of the size of  $\text{Odd}(H_n)$ . We show in section IV that with our definition for the FW sequence roughly  $N$  iterations are needed to arrive at a nonrelativistic hamiltonian that is even to order  $(|\vec{p}|/m)^{(3^N-1)}$ .

To start the FW sequence, choose

$$H_0 = m_0(\gamma^0 + \mathcal{P}_0 + \mathcal{O}_0 + \mathcal{E}_0) \quad , \quad (9)$$

where  $m_0 := m$  and the terms  $\mathcal{P}_0$ ,  $\mathcal{O}_0$ , and  $\mathcal{E}_0$  are defined in Eq. (7). This decomposition of  $H_0$  into four parts has the following useful properties: (i)  $\mathcal{P}_0$  and  $\mathcal{O}_0$  are odd; (ii)  $\mathcal{E}_0$  is even; (iii)  $\mathcal{O}_0$  and  $\mathcal{E}_0$  are first order in parameters for Lorentz violation, so products of these quantities can be neglected; and (iv)  $\mathcal{P}_0^2$  is proportional to the  $4 \times 4$  Dirac identity matrix, with proportionality coefficient  $t_0^2 = |\vec{p}|^2/m^2$ . We choose for the initial FW transformation the hermitian matrix  $S_0$  defined by

$$iS_0 := \frac{1}{2m_0}\gamma^0[\text{Odd}(H_0)] = \frac{1}{2}\gamma^0(\mathcal{P}_0 + \mathcal{O}_0) \quad . \quad (10)$$

This choice ensures that the odd part of  $\exp[\text{ad}(iS_0)]H_0$  is smaller than the odd part of  $H_0$ .

Our FW sequence is then defined iteratively by

$$H_{n+1} := e^{iS_n} H_n e^{-iS_n} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{[iS_n, [iS_n, \dots [iS_n, H_0] \dots]]}_{k \text{ commutations with } iS_n} = \exp[\text{ad}(iS_n)]H_n \quad (11)$$

and

$$iS_{n+1} := \frac{1}{2m_{n+1}}\gamma^0 \times \text{Odd}(H_{n+1}) \quad . \quad (12)$$

Note that

$$H_{n+1} = \left\{ \prod_{k=0}^n \exp[\text{ad}(iS_k)] \right\} H_0 \quad , \quad (13)$$

where the product represents map composition.

In the next section, we find that each  $H_{n+1}$  can be written in the form

$$H_{n+1} = m_{n+1}(\gamma^0 + \mathcal{P}_{n+1} + \mathcal{O}_{n+1} + \mathcal{E}_{n+1}) \quad , \quad (14)$$

where the decomposition has the following useful properties: (i)  $\mathcal{P}_{n+1}$  and  $\mathcal{O}_{n+1}$  are odd; (ii)  $\mathcal{E}_{n+1}$  is even; (iii)  $\mathcal{O}_{n+1}$  and  $\mathcal{E}_{n+1}$  are first order in parameters for Lorentz violation; and (iv)  $\mathcal{P}_{n+1}^2$  is proportional to the identity matrix, with proportionality coefficient  $t_{n+1}^2$  determined by  $t_n^2$ . The existence of a decomposition of the form (14) for arbitrary  $n$ , as well as the case (9) above, is a key feature making it feasible to calculate the quantum particle hamiltonian.

#### IV. CALCULATION OF THE FW SEQUENCE

To calculate the FW sequence defined in the previous section, the explicit form is needed of the operator  $\exp[\text{ad}(iS_n)]$  connecting  $H_n$  to  $H_{n+1}$  according to Eq. (11). Although  $\text{ad}(iS_n)H_n$  can be obtained directly using the properties of the Dirac matrices, calculation of  $\exp[\text{ad}(iS_j)]H_n$  is more challenging because it is defined by an infinite series. To address this issue, we adopt the following approach: regard  $\text{ad}(iS_n)$  as a linear map on a suitable vector space  $\mathcal{V}_n$  containing both  $H_n$  and  $H_{n+1}$ , and find a matrix expression of this map that can be exponentiated.

The first step in implementing this approach is to define  $\mathcal{V}_n$  for each  $n$ . It is convenient to introduce  $\mathcal{V}_n$  as the span of a set of basis vectors  $\mathcal{B}_n$ , defined in terms of the operators  $\gamma^0$ ,  $\mathcal{P}_n$ ,  $\mathcal{O}_n$ ,  $\mathcal{E}_n$  determining  $H_n$  together with the particular combinations of these four operators that determine  $\text{ad}(iS_n)H_n$  and thus also  $H_{n+1}$ . For each  $n$ , we define the ordered set

$$\mathcal{B}_n := \{\gamma^0, \mathcal{P}_n, \mathcal{O}_n, \mathcal{P}_n\{\mathcal{P}_n, \mathcal{O}_n\}, \gamma^0[\mathcal{P}_n, \mathcal{E}_n], \mathcal{E}_n, \gamma^0\{\mathcal{P}_n, \mathcal{O}_n\}, \mathcal{P}_n[\mathcal{P}_n, \mathcal{E}_n]\} \quad . \quad (15)$$

The eight-dimensional vector space  $\mathcal{V}_n$  is formally defined as the real span of this set, so the elements of  $\mathcal{B}_n$  by definition form a (linearly independent) basis. One advantage of this vector space is its relatively small dimensionality, which makes it susceptible to practical calculation. We can thus specify a vector  $V \in \mathcal{V}_n$  by eight components  $V_1, \dots, V_8$ :

$$\begin{aligned} V &:= V_1\gamma^0 + V_2\mathcal{P}_n + V_3\mathcal{O}_n + V_4\mathcal{P}_n\{\mathcal{P}_n, \mathcal{O}_n\} + V_5\gamma^0[\mathcal{P}_n, \mathcal{E}_n] + V_6\mathcal{E}_n + V_7\gamma^0\{\mathcal{P}_n, \mathcal{O}_n\} + V_8\mathcal{P}_n[\mathcal{P}_n, \mathcal{E}_n] \\ &\leftrightarrow (V_1, \dots, V_8) \quad . \end{aligned} \quad (16)$$

For example,  $H_n \leftrightarrow m_n(1, 1, 1, 0, 0, 1, 0, 0)$ .

The reader is warned to avoid confusing the properties of the elements (15) as a basis for the vector space  $\mathcal{V}_n$  with their possible relationships when viewed as operators on the Hilbert space of wave functions. For example, the calculations below hold even if certain basis elements vanish as operators. Note also that for different  $n$  the corresponding vector spaces  $\mathcal{V}_n$  differ *a priori*. However, since both  $H_n \in \mathcal{V}_n$  and  $H_{n+1} \in \mathcal{V}_n$ , the vector space  $\mathcal{V}_n$  is invariant under the action of  $\exp[\text{ad}(iS_n)]$ , which means  $\mathcal{V}_n \supseteq \mathcal{V}_{n+1}$  for all  $n$ .

With the above notation, we can present the results of a direct calculation of  $\text{ad}(iS_n)V$  for  $V \in \mathcal{V}_n$  performed using the properties of the Dirac matrices:

$$\text{ad}(iS_n)V \leftrightarrow (t_n^2 V_2, -V_1, -V_1, -V_7, \frac{1}{2}V_6 + t_n^2 V_8, 0, \frac{1}{2}V_2 + \frac{1}{2}V_3 + t_n^2 V_4, -V_5) \quad . \quad (17)$$

In this expression,  $t_n^2$  is determined iteratively from  $t_{n-1}^2$  through the relation

$$t_{n+1}^2 = \left( \frac{\cos t_n - \frac{\sin t_n}{t_n}}{\cos t_n + t_n \sin t_n} \right)^2 t_n^2 \quad . \quad (18)$$

Here and in what follows, we define functions of  $t_n$  through their power-series expressions. All relevant functions of  $t_n$  implicitly involve only powers of  $t_n^2$  (and hence powers of  $t_0^2 = |\vec{p}|^2/m^2$ ), so it suffices to define  $t_n^2$ . Note that  $t_{n+1} \sim t_n^3$  to leading order in  $t_n$ , so  $t_n \sim t_0^{(3^n)}$ . This means that  $t_n$  rapidly approaches zero if  $t_0 \ll 1$ , which ultimately is the reason for the rapid convergence of our FW sequence.

With respect to the basis  $\mathcal{B}_n$ , the matrix map of  $\text{ad}(iS_n)$  can be extracted from Eq. (17) and is given by

$$\text{ad}(iS_n) \leftrightarrow \begin{pmatrix} 0 & t_n^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & t_n^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & t_n^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad . \quad (19)$$

The exponential of this matrix can be found in closed form, but its detailed expression is unimportant. It can be used to calculate  $\exp[\text{ad}(iS_n)]H_n$ , which allows us to express  $H_{n+1}$  in terms of  $H_n$  according to Eq. (14) with

$$\begin{aligned}
m_{n+1} &= (\cos t_n + t_n \sin t_n) m_n \quad , \quad m_{n+1} \mathcal{P}_{n+1} = \left( \cos t_n - \frac{\sin t_n}{t_n} \right) m_n \mathcal{P}_n \quad , \\
m_{n+1} \mathcal{O}_{n+1} &= \left( \cos t_n - \frac{\sin t_n}{t_n} \right) m_n \mathcal{O}_n + \frac{1}{2t_n^2} \left( \frac{\sin t_n}{t_n} - t_n \sin t_n - \cos t_n \right) m_n \mathcal{P}_n \{ \mathcal{P}_n, \mathcal{O}_n \} + \frac{\sin t_n}{2t_n} m_n \gamma^0 [ \mathcal{P}_n, \mathcal{E}_n ] \quad , \\
m_{n+1} \mathcal{E}_{n+1} &= m_n \mathcal{E}_n + \left( \frac{1}{2} \cos t_n \right) m_n \gamma^0 \{ \mathcal{P}_n, \mathcal{O}_n \} + \left( \frac{\cos t_n - 1}{2t_n^2} \right) m_n \mathcal{P}_n [ \mathcal{P}_n, \mathcal{E}_n ] \quad .
\end{aligned} \tag{20}$$

A measure of the convergence of the FW sequence can be introduced using  $t_n \sim t_0^{(3^n)}$ . In terms of a suitable matrix norm,  $\|\text{Odd}(H_{n+1})\| \sim t_n^2 \|\text{Odd}(H_n)\| + t_n \|\mathcal{E}_0\| \sim t_0^{2(3^n)} \|\text{Odd}(H_n)\| + t_0^{(3^n)} \|\mathcal{E}_0\|$ . Thus, as  $n$  grows  $\|\text{Odd}(H_n)\|$  rapidly approaches zero as  $(|\vec{p}|/m)^{3^n}$ . Even a relatively small value of  $n$  can therefore produce a good approximation to the quantum particle hamiltonian.

## V. NONRELATIVISTIC QUANTUM HAMILTONIAN

The quantum particle hamiltonian  $h_{\text{rel}}$  and its nonrelativistic quantum limit  $h$  are generated in the limit of the FW sequence studied in the last section. Next, we demonstrate how to obtain these using simple matrix multiplication, and we explicitly present  $h_{\text{rel}}$  and  $h$  to order  $t_0^3$ .

The calculation at the  $k$ th-iteration level in the FW sequence requires obtaining the composite map  $\prod_{n=0}^k \exp[\text{ad}(iS_n)]$ . For each  $n$  in the FW sequence, the matrix  $\text{ad}(iS_n)$  and the action of  $\exp[\text{ad}(iS_n)]$  are given with respect to the basis  $\mathcal{B}_n$ . Since in general the vector space  $\mathcal{V}_n$  varies with  $n$ , immediate calculation of  $\prod_{n=0}^k \exp[\text{ad}(iS_n)]$  by matrix multiplication is inappropriate. Instead, we first obtain the components of each matrix with respect to the special basis  $\mathcal{B}_0$ . Ordinary matrix multiplication can then be used to derive  $\prod_{n=0}^k \exp[\text{ad}(iS_n)]$ .

The matrix for each map  $\exp[\text{ad}(iS_n)]$  can be expressed in terms of  $t_n$ . Explicitly, the nonzero entries for  $\exp[\text{ad}(iS_n)]$  with respect to the basis  $\mathcal{B}_0$  are:

$$\exp[\text{ad}(iS_n)] \leftrightarrow \begin{pmatrix} c_n + t_n^2 s_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_n - s_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_n - s_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (s_n - t_n^2 s_n - c_n)/2t_0^2 & -t_n^2 s_n & 0 & 0 & -t_n s_n/t_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t_n^2 s_n & t_n s_n/2t_0 & 0 & t_0 t_n s_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t_n c_n/2t_0 & t_0 t_n c_n & 0 & 0 & c_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -t_n c_n/t_0 & (c_n - 1)/2t_0^2 & 0 & c_n & 0 \end{pmatrix} \quad , \tag{21}$$

where we have defined  $c_n := \cos t_n \approx 1 - \frac{1}{2}t_n^2$  and  $s_n := \sin t_n/t_n \approx 1 - \frac{1}{6}t_n^2$ .

Since  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $\exp[\text{ad}(iS_n)]$  becomes a diagonal matrix with entries  $(1, 0, 0, 0, 0, 1, 1, 1)$  in this limit. The product  $\prod_{n=0}^k \exp[\text{ad}(iS_n)]$  therefore converges as  $k \rightarrow \infty$ , so the limiting FW sequence giving the quantum particle hamiltonian indeed exists. It can be shown that

$$\prod_{j=0}^{\infty} \exp[\text{ad}(iS_j)] \leftrightarrow \begin{pmatrix} \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\gamma} & [t_0^2] & 0 & 0 & [1 - \frac{1}{2}t_0^2] & 0 & 0 \\ 0 & 0 & 0 & 0 & [-1 + \frac{1}{2}t_0^2] & -\frac{1}{2\gamma(\gamma+1)} & 0 & [1 - \frac{1}{2}t_0^2] & 0 \end{pmatrix} \quad , \tag{22}$$

where  $\gamma := \sqrt{1 + t_0^2}$  is the usual relativistic gamma factor. Since the limiting FW hamiltonian is obtained by applying this matrix to  $H_0 \leftrightarrow (1, 1, 1, 0, 0, 1, 0, 0)$ , the entries in brackets are irrelevant and so we have evaluated them only to order  $t_0^2$ . Thus, we find the limiting FW hamiltonian to be

$$\tilde{H} = \gamma m_0 \gamma^0 + m_0 \mathcal{E}_0 + \frac{m_0}{2\gamma} \gamma^0 \{ \mathcal{P}_0, \mathcal{O}_0 \} - \frac{m_0}{2\gamma(\gamma+1)} \mathcal{P}_0 [ \mathcal{P}_0, \mathcal{E}_0 ] \quad , \tag{23}$$

an expression that is accurate to all orders in  $t_0$ . Substitution from Eq. (7) yields the explicit form

$$\begin{aligned}
\tilde{H} = & \gamma m \gamma^0 + \left\{ a_0 - m e_0 - m(c_{0j} + c_{j0}) \frac{p^j}{m} \right\} + \left\{ -\frac{m c_{00}}{\gamma} + (a_j - m e_j) \frac{p^j}{\gamma m} - m(c_{jk} - \eta_{jk} c_{00}) \frac{p^j p^k}{\gamma m^2} \right\} \gamma^0 \\
& + \left\{ -(m d_{j0} + \frac{1}{2} \varepsilon^{kl}{}_j H_{kl}) + \left[ -\frac{b_0 \eta_{jk}}{\gamma} + m \varepsilon^{lm}{}_j (\frac{1}{2} g_{lmk} - \eta_{km} g_{l00}) \right] \frac{p^k}{m} \right. \\
& + \left[ m(d_{0l} + d_{l0}) - \frac{(\gamma-1)m^2}{p^2} (m d_{l0} + \frac{1}{2} \varepsilon^{mn}{}_l H_{mn}) \right] \eta_{jk} \frac{p^l p^k}{\gamma m^2} + \left[ \frac{(\gamma-1)m^2}{2p^2} m \varepsilon^{nq}{}_l g_{nqk} \right] \eta_{jm} \frac{p^k p^l p^m}{\gamma m^3} \left. \right\} \gamma^5 \gamma^j \\
& + \left\{ [-b_j - \frac{1}{2} m \varepsilon^{kl}{}_j g_{kl0}] \frac{1}{\gamma} + [\varepsilon^l{}_{kj} H_{0l} + m(d_{jk} - \eta_{jk} d_{00})] \frac{p^k}{\gamma m} \right. \\
& + \left[ m \varepsilon^{m}{}_{lj} (g_{m0k} + g_{mk0}) + \frac{(\gamma-1)m^2}{p^2} \eta_{jl} (b_k + \frac{1}{2} m \varepsilon^{mn}{}_k g_{mn0}) \right] \frac{p^k p^l}{\gamma m^2} \\
& + \left. \left[ -\frac{(\gamma-1)m^2}{p^2} m(d_{kl} - \eta_{kl} d_{00}) \right] \eta_{jm} \frac{p^k p^l p^m}{\gamma m^3} \right\} \gamma^5 \gamma^0 \gamma^j \quad . \quad (24)
\end{aligned}$$

This equation gives the FW form of the relativistic quantum hamiltonian for a four-component fermion.

Certain limiting forms of the above expression are directly relevant to experiment. For applications involving relativistic two-component particles, such as the analysis of muon storage-ring experiments [14], it suffices to retain only the upper left block  $h_{\text{rel}}$  of  $\tilde{H}$ :

$$\begin{aligned}
h_{\text{rel}} = & \gamma m + (a_0 - \frac{m c_{00}}{\gamma} - m e_0) + \left[ a_j - \gamma m(c_{0j} + c_{j0}) - m e_j \right] \frac{p^j}{\gamma m} - m(c_{jk} - \eta_{jk} c_{00}) \frac{p^j p^k}{\gamma m^2} \\
& + \left\{ \left[ -\frac{1}{\gamma} b_j + m d_{j0} + \frac{1}{2} \varepsilon^{kl}{}_j H_{kl} - \frac{1}{2\gamma} m \varepsilon^{kl}{}_j g_{kl0} \right] \right. \\
& + \left[ \eta_{jk} b_0 + m(d_{jk} - \eta_{jk} d_{00}) + \varepsilon^l{}_{kj} H_{0l} - \gamma m \varepsilon^{lm}{}_j (\frac{1}{2} g_{lmk} - \eta_{km} g_{l00}) \right] \frac{p^k}{\gamma m} \\
& + \left[ \frac{(\gamma-1)m^2}{p^2} (b_k + m d_{k0} + \frac{1}{2} \varepsilon^{mn}{}_k H_{mn} + \frac{1}{2} m \varepsilon^{mn}{}_k g_{mn0}) \eta_{jl} \right. \\
& \quad \left. - m(d_{0k} + d_{k0}) \eta_{jl} + m \varepsilon^m{}_{lj} (g_{m0k} + g_{mk0}) \right] \frac{p^k p^l}{\gamma m^2} \\
& + \left. \frac{(\gamma-1)m^2}{p^2} \left[ -m(d_{kl} - \eta_{kl} d_{00}) - \frac{1}{2} m \varepsilon^{nq}{}_l g_{nqk} \right] \eta_{jm} \frac{p^k p^l p^m}{\gamma m^3} \right\} \sigma^j \quad . \quad (25)
\end{aligned}$$

This is the quantum particle hamiltonian associated with the original Lorentz-violating theory.

For many low-energy applications, including analyses of high-precision atomic experiments [5–8,15], only nonrelativistic and subleading relativistic terms in the quantum particle hamiltonian are needed. To third order in  $|\vec{p}|/m$ , the nonrelativistic quantum hamiltonian  $h$  for the two-component fermion is

$$\begin{aligned}
h = & m + \frac{p^2}{2m} \\
& + (a_0 - m c_{00} - m e_0) + (-b_j + m d_{j0} - \frac{1}{2} m \varepsilon_{jkl} g_{kl0} + \frac{1}{2} \varepsilon_{jkl} H_{kl}) \sigma^j + [-a_j + m(c_{0j} + c_{j0}) + m e_j] \frac{p_j}{m} \\
& + [b_0 \delta_{jk} - m(d_{kj} + d_{00} \delta_{jk}) - m \varepsilon_{klm} (\frac{1}{2} g_{mlj} + g_{m00} \delta_{jl}) - \varepsilon_{jkl} H_{l0}] \frac{p_j}{m} \sigma^k + [m(-c_{jk} - \frac{1}{2} c_{00} \delta_{jk})] \frac{p_j p_k}{m^2} \\
& + \left\{ [m(d_{0j} + d_{j0}) - \frac{1}{2} (b_j + m d_{j0} + \frac{1}{2} m \varepsilon_{jmn} g_{mn0} + \frac{1}{2} \varepsilon_{jmn} H_{mn})] \delta_{kl} \right. \\
& \quad \left. + \frac{1}{2} (b_l + \frac{1}{2} m \varepsilon_{lmn} g_{mn0}) \delta_{jk} - m \varepsilon_{jlm} (g_{m0k} + g_{mk0}) \right\} \frac{p_j p_k}{m^2} \sigma^l \\
& + \frac{1}{2} (a_j \delta_{kl} - m e_j \delta_{kl}) \frac{p_j p_k p_l}{m^3} \\
& + \frac{1}{2} [(-b_0 \delta_{jm} + m d_{mj} + \varepsilon_{jmn} H_{n0}) \delta_{kl} + (-m d_{jk} - \frac{1}{2} m \varepsilon_{knp} g_{npj}) \delta_{lm}] \frac{p_j p_k p_l \sigma^m}{m^3} \quad . \quad (26)
\end{aligned}$$

Note that the form of Eq. (23) includes all even elements of the basis set  $\mathcal{B}_0$ . This means that all possible combinations of the parameters for Lorentz violation are already contained in Eq. (26). Higher-order corrections to the nonrelativistic hamiltonian involve only products of these combinations with powers of  $|\vec{p}|^2/m^2$ . One interesting implication of this result is that nonrelativistic experiments with single free fermions (or fermions in weak external fields) can at most be sensitive to the particular linear combinations of parameters for Lorentz violation appearing in Eq. (26). Disentangling individual parameters requires a different class of experiment.

As a final remark, note that our methods can also be used to obtain the nonrelativistic quantum hamiltonian  $\bar{h}$  for the antifermion. The result for  $\bar{h}$  can be expressed in the same form as Eq. (26), with the substitutions  $a_\mu \rightarrow \bar{a}_\mu = -a_\mu$ ,  $b_\mu \rightarrow \bar{b}_\mu = +b_\mu$ ,  $c_{\mu\nu} \rightarrow \bar{c}_{\mu\nu} = +c_{\mu\nu}$ ,  $d_{\mu\nu} \rightarrow \bar{d}_{\mu\nu} = -d_{\mu\nu}$ ,  $e_\mu \rightarrow \bar{e}_\mu = -e_\mu$ ,  $f_\mu \rightarrow \bar{f}_\mu = -f_\mu$ ,  $g_{\lambda\mu\nu} \rightarrow \bar{g}_{\lambda\mu\nu} = +g_{\lambda\mu\nu}$ ,  $H_{\mu\nu} \rightarrow \bar{H}_{\mu\nu} = -H_{\mu\nu}$ . This result is useful for experiments testing Lorentz symmetry with antimatter.

## VI. ACKNOWLEDGMENTS

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